Selected issues in the theory of nonlinear oscillations

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Abstract. The paper presents results concerning the theory of oscillations in the field of linear extensions of dynamical systems. An overview of the basic results was done, the direction of research was outlined and the results obtained were given in this regard.

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1. Introduction

In the theory of non-linear multi-frequency oscillations several questions related to the research of invariant tori of autonomous systems of differential equations arise. One of the important issues is to maintain invariant tori for small disturbances, as well as the behaviour of the solutions of an equation on the same tori and within their neighbourhood. Together with deep research in this direction (see [2, 10, 11]), there is a number of problems that currently can not be fully resolved. This article is a review of some of the problems that arise in the use of Lyapunov functions in the theory of linear extensions of dynamical systems on the torus. Similar research can be found in [1, 9, 14, 12, 15].

Let us consider the system of differential equations

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x + f(\varphi), \tag{1}$$

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where $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_m)$, $x = (x_1, x_2, \ldots, x_n)$, the vector function $a(\varphi) = (a_1(\varphi), a_2(\varphi), \ldots, a_m(\varphi))$ defined for all $\varphi \in \mathbb{R}^m$ is real, continuous and periodic with respect to each variable φ_j with the period 2π. Usually, it is said that the function $a(\varphi)$ is defined on the m-dimensional torus T_m, thus it belongs to the space of continuous functions $C(T_m)$. It is assumed that the Cauchy problem $\frac{d\varphi}{dt} = a(\varphi)$, $\varphi|_{t=0} = \varphi$ has a unique solution, denoted by $\varphi_t(\varphi)$ (see [2, 10]). Since the periodic function $a(\varphi)$ is bounded, the solutions $\varphi_t(\varphi)$ will always be defined on the whole real axis $\mathbb{R}$. The matrix $A(\varphi)$ in the system (1) is a square matrix whose elements are real, continuous and 2π-periodic functions, i.e. $A(\varphi) \in C(T_m)$, the vector function $f(\varphi) \in C(T_m)$. The system (1) is used to be called a linear extension of a dynamical system on the torus. Together with the system (1), we will consider the corresponding homogeneous system in the following form

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x. \tag{2}$$

Recall the definition of space $C'(T_m; a)$ of an invariant torus of the system (1) and the definition of the Green-Samoilenko function for the problem with an invariant torus $G_0(\tau, \varphi)$ for the system (2).

Definition 1.1. *$C'(T_m; a)$ is a subspace $C(T_m)$ of continuous functions $\Phi(\varphi)$ such that the superposition $\Phi(\varphi_t(\varphi))$ is continuously differentiable with respect to the variable t, $t \in \mathbb{R}$, whereas $\frac{d\Phi(\varphi_t(\varphi))}{dt}|_{t=0} \overset{df}{=} \dot{\Phi}(\varphi)$.*

Definition 1.2 (see [5])**.** *We say that the system (1) has an invariant torus, defined by the equality $x = u(\varphi)$, if $u(\varphi) \in C'(T_m; a)$ and an identity: $\dot{u}(\varphi) \equiv A(\varphi)u(\varphi) + f(\varphi)$ for every $\varphi \in T_m$, is satisfied.*

Denoting a fundamental matrix of solutions normalised for $t = \tau$, i.e. $\Omega^t_\tau|_{t=\tau} = I_n$, where I_n is an n-dimensional identity matrix of the linear system

$$\frac{dx}{dt} = A(\varphi_t(\varphi))x, \tag{3}$$

with $\Omega^t_\tau(\varphi)$ ($\Omega^t_\tau(\varphi) = \Omega^t_\tau(\varphi; A)$), let us recall the definition of the Green-Samoilenko function [2].

Definition 1.3. *If there exists an n-dimensional square matrix $C(\varphi) \in C(T_m)$, such that the function*

$$G_0(\tau, \varphi) = \begin{cases} \Omega^0_\tau(\varphi) C(\varphi_\tau(\varphi)), & \tau \leqslant 0, \\ \Omega^0_\tau(\varphi) \left[C(\varphi_\tau(\varphi)) - I_n\right], & \tau > 0, \end{cases} \tag{4}$$

satisfies the estimate

$$\|G_0(\tau, \varphi)\| \leqslant K e^{-\gamma|\tau|} \quad \forall \tau \in \mathbb{R} \quad \forall \varphi \in T_m, \tag{5}$$

where K, γ are positive constants independent of τ and φ, then the function (4) is called the Green-Samoilenko function for the problem with an invariant torus of the system (2).

Remark 1.4. If there exists a Green-Samoilenko function (4) satisfying the estimate (5), then the system (1) has an invariant torus in the form of

$$x = u(\varphi) = \int_{-\infty}^{+\infty} G_0(\tau, \varphi) f(\varphi_\tau(\varphi)) d\tau, \tag{6}$$

for all fixed functions $f(\varphi) \in C(T_m)$.

Remark 1.5. (See [10]). To establish the Green-Samoilenko function, the estimate (5) can be weakened – it is enough to require that the integral

$$\int_{-\infty}^{+\infty} \|G_0(\tau, \varphi)\| d\tau \tag{7}$$

converges uniformly with respect to variables $\varphi \in T_m$, and then the equation (6) determines the invariant torus (1), whereas a problem with the study of the function smoothness (6) with respect to φ appears.

There are examples of systems (2) that have a function in the form of (4) for which the estimate (5) is not satisfied, and the integral (7) is uniformly convergent with respect to the variables φ. However, in the examples considered there was another Green-Samoilenko function that satisfied the estimate (5). The question arises whether or not, if there is a function in the form of (4), for which the integral (7) is uniformly convergent, there always exists a function (possibly different) in the form of (4) for which the estimate (5) is satisfied. This question still remains unanswered.

Remark 1.6. Since

$$\varphi_t(\varphi_z(\varphi)) \equiv \varphi_{t+z}(\varphi),$$

the fulfilment of the inequality (5) for the function (4) is equivalent to the fulfilment of the following estimate

$$\|G_t(0, \varphi)\| \leqslant K e^{-\gamma|t|} \qquad \forall\, t \in \mathbb{R} \;\; \forall \varphi \in T_m$$

for the auxiliary function

$$G_t(0, \varphi) = \begin{cases} \Omega_0^t(\varphi) C(\varphi), & t \geqslant 0, \\ \Omega_0^t(\varphi) \left[C(\varphi) - I_n\right], & t < 0, \end{cases} \tag{8}$$

with the same positive constants K, γ.

Remark 1.7. If there exists a Green-Samoilenko function (4) with an estimate (5), the following function

$$G_t(\tau, \varphi) = \begin{cases} \Omega_\tau^t(\varphi) C(\varphi_\tau(\varphi)), & \tau \leqslant t, \\ \Omega_\tau^t(\varphi) \left[C(\varphi_\tau(\varphi)) - I_n\right], & \tau > t, \end{cases} \tag{9}$$

is a Green's function for a problem with bounded solutions of the system (3), i.e. the heterogeneous system

$$\frac{dx}{dt} = A(\varphi_t(\varphi))x + f(t) \tag{10}$$

has the bounded solution $x = \int\limits_{-\infty}^{+\infty} G_t(\tau, \varphi) f(\tau) d\tau$ for each function $f(t)$ being continuous and bounded on $\mathbb{R}$.

Corollary 1.8. *If a heterogeneous linear system (10) has no bounded solution on $\mathbb{R}$ for a certain parameter value φ and a certain function $f(t)$ continuous and bounded on $\mathbb{R}$, then a Green-Samoilenko function (4) does not exist.*

2. Overview of the main results

The system (2) that has a unique Green-Samoilenko function with an estimate (5) is used to be called **regular**, and if it is known that there is at least one such function (4), then the system (2) is **weakly regular**, and in the case where there is infinitely many different of such functions, the system (2) is **strictly weakly regular.**

Let us recall that $\langle y, \bar{y}\rangle = \sum\limits_{j=1}^{n} y_j \bar{y}_j$ represents the inner product in $\mathbb{R}^n$, $\|y\| = \sqrt{\langle y, y\rangle}$ is a norm of the vector y in $\mathbb{R}^n$, $\langle Sy, y\rangle$ is a quadratic form associated with the symmetric matrix S.

In the book [10] on page 124 the following theorem is formulated.

Theorem 2.1. *Suppose that the following quadratic form exists*

$$W = \langle S(\varphi)y, y\rangle, \qquad y \in \mathbb{R}^n, \tag{11}$$

associated with a symmetric matrix $S(\varphi) \in C'(T_m; a)$, whose derivative with respect to the system of equations

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dy}{dt} = -A^T(\varphi)y, \tag{12}$$

is positive definite, thus

$$\dot{W} = \left\langle \left[\dot{S}(\varphi) - S(\varphi)A^T(\varphi) - A(\varphi)S(\varphi)\right] y, y \right\rangle \geqslant \|y\|^2, \qquad y \in \mathbb{R}^n. \tag{13}$$

Under the above assumptions the system (2) is weakly regular. If in addition we assume that

$$\det S(\varphi) \neq 0 \quad \forall \varphi \in T_m, \tag{14}$$

then the system (2) is regular and in the structure of the Green-Samoilenko function (4) the matrix $C(\varphi)$ is a projection matrix for which the following identities are satisfied

$$C^2(\varphi) \equiv C(\varphi), \quad C(\varphi_t(\varphi)) \equiv \Omega_0^t(\varphi)C(\varphi)\Omega_t^0(\varphi) \quad \forall \varphi \in T_m,\, t \in \mathbb{R}. \tag{15}$$

If the condition (14) is not satisfied and there exist such values $\varphi = \varphi_0 \in T_m$ for which the following the equality is satisfied $\det S(\varphi_0) = 0$ *then the system (2) is strictly weakly regular and each of the matrices $C(\varphi)$ does not satisfy any of the identities (15).*

In the book [11] on page 130 identities are examined in detail.

It is easy to see (by making the change of variables $y = S^{-1}(\varphi)x$) that the following remark is true.

Remark 2.2. If the symmetric matrix $S(\varphi) \in C'(T_m; a)$ is non-degenerated in the inequality (13), then for the matrix $\bar{S}(\varphi) = -S^{-1}(\varphi)$ the following inequality is satisfied

$$\langle \left[\dot{\bar{S}}(\varphi) + \bar{S}(\varphi)A(\varphi) + A^T(\varphi)\bar{S}(\varphi) \right] x, x \rangle \geqslant \gamma \|x\|^2, \qquad x \in \mathbb{R}^n, \qquad \gamma = \mathrm{const} > 0,$$

and which means that the derivative of the non-degenerated quadratic form

$$-\langle S^{-1}(\varphi)x, x \rangle = \langle \bar{S}(\varphi)x, x \rangle$$

with respect to the system (2) is positive definite.

A converse theorem is established (see [10] on page 123).

Theorem 2.3. *If the system (2) has the Green-Samoilenko function (4) with the estimate (5), then there exist symmetric matrices $S(\varphi)$ that satisfy the inequality (13). Some of these matrices can be expressed in the following form*

$$\begin{aligned} S(\varphi) &= 2[S_1(\varphi) - S_2(\varphi)], \\ S_1(\varphi) &= \int_{-\infty}^{0} \Omega_\tau^0(\varphi) C(\varphi_\tau(\varphi)) \left[\Omega_\tau^0(\varphi) C(\varphi_\tau(\varphi)) \right]^T d\tau, \\ S_2(\varphi) &= \int_{0}^{+\infty} \Omega_\tau^0(\varphi) \left[C(\varphi_\tau(\varphi)) - I_n \right] \left\{ \Omega_\tau^0(\varphi) \left[C(\varphi_\tau(\varphi)) - I_n \right] \right\}^T d\tau. \end{aligned}$$

Remark 2.4. The form of matrices $S(\varphi)$ was generalized (the proof can be found in [7]):

$$\begin{aligned} S(\varphi) = &\int_{-\infty}^{0} \Omega_\tau^0(\varphi) C(\varphi_\tau(\varphi)) H_1(\varphi_\tau(\varphi)) \left[\Omega_\tau^0(\varphi) C(\varphi_\tau(\varphi)) \right]^T d\tau - \\ &- \int_{0}^{+\infty} \Omega_\tau^0(\varphi) \left[C(\varphi_\tau(\varphi)) - I_n \right] H_2(\varphi_\tau(\varphi)) \left\{ \Omega_\tau^0(\varphi) \left[C(\varphi_\tau(\varphi)) - I_n \right] \right\}^T d\tau, \end{aligned} \tag{16}$$

where $H_i(\varphi) \in C(T_m)$ are any symmetric matrices, satisfying the inequality

$$\langle H_i(\varphi)y, y \rangle \geqslant 2\|y\|^2, \qquad i = 1, 2.$$

Simultaneously, if the system (2) is regular, then the inequality (14) is satisfied for any symmetric matrix $S(\varphi) \in C'(T_m; a)$ that satisfies the condition (13).

Regarding to the norm of a matrix A defined as $\|A\| = \max\limits_{\|x\|=1} \|Ax\|$, it has been proven in [13] (pp. 1685–1686) that from the inequality

$$\|\Omega_0^\Delta(\varphi)\| < 1 \quad \forall \varphi \in T_m \tag{17}$$

for any fixed value $\Delta > 0$ we obtain an estimation

$$\|\Omega_\tau^t(\varphi)\| \leqslant Ke^{-\gamma(t-\tau)}, \quad t, \tau \in \mathbb{R}, \quad \tau \leqslant t, \tag{18}$$

where K and γ are certain positive constants. Hence it follows the theorem.

Theorem 2.5. *Let the constant $\Delta > 0$ such that $\|\Omega_0^\Delta(\varphi)\| < 1$ for every $\varphi \in T_m$ exist. Then the system (2) is regular and the Green–Samoilenko function has the form*

$$G_0(\tau, \varphi) = \begin{cases} \Omega_\tau^0(\varphi), & \tau \leqslant 0, \\ 0, & \tau > 0. \end{cases}$$

Another proof of the fact that the inequality (17) implies the estimation (18) has been proposed in [4] (pp. 92–93). Namely, a symmetric matrix was considered in the form

$$\int_t^{\Delta+t} \left(\Omega_t^\sigma(\varphi)\right)^T \Omega_t^\sigma(\varphi)\, d\sigma = S(\varphi_t(\varphi)). \tag{19}$$

Whereas the quadratic form $\langle S(\varphi_t(\varphi)x, x\rangle$ is positive definite and its derivative with respect to the system (3) is negative definite.

Example 2.6. Let us examine the regularity of the system

$$\frac{d\varphi}{dt} = 1 + \varepsilon\cos\varphi, \quad 0 < \varepsilon < 1, \quad \frac{dx}{dt} = (\cos\varphi)x.$$

The regularity of this system is equivalent to the regularity of the following system

$$\frac{d\varphi}{dt} = 1, \quad \frac{dx}{dt} = \frac{\cos\varphi}{1 + \varepsilon\cos\varphi}x,$$

for this system the value of $\Omega_0^{2\pi}(\varphi)$ will be less than 1, namely

$$\begin{aligned}
\Omega_0^{2\pi}(\varphi) &= \exp\left\{\int_0^{2\pi} \frac{\cos(t+\varphi)}{1+\varepsilon\cos(t+\varphi)}\, dt\right\} = \\
&= \exp\left\{\int_0^{\pi} \frac{\cos(t+\varphi)}{1+\varepsilon\cos(t+\varphi)}\, dt + \int_\pi^{2\pi} \frac{\cos(t+\varphi)}{1+\varepsilon\cos(t+\varphi)}\, dt\right\} = \\
&= \exp\left\{\int_0^{\pi} \left[\frac{\cos(t+\varphi)}{1+\varepsilon\cos(t+\varphi)} - \frac{\cos(t+\varphi)}{1-\varepsilon\cos(t+\varphi)}\right] dt\right\} = \\
&= \exp\left\{\int_0^{\pi} \frac{-2\varepsilon\cos^2(t+\varphi)}{1-\varepsilon^2\cos^2(t+\varphi)}\, dt\right\} < 1.
\end{aligned}$$

In this way the system considered is regular for the parameter value $\varepsilon \in (0, 1)$. If $\varepsilon = 0$, the system is not regular.

If we now consider the matrix

$$\int_{T_1(\varphi)}^{T_2(\varphi)} \left(\Omega_0^\sigma(\varphi)\right)^T \Omega_0^\sigma(\varphi)\, d\sigma = S(\varphi),$$

which is a generalization of (19), then calculating the derivative of a quadratic form $\langle S(\varphi)x, x\rangle$ with respect to the system (2) we obtain the following theorem.

Theorem 2.7. *Let the two scalar functions $T_1(\varphi), T_2(\varphi) \in C'(T_m; a)$ exist such that the quadratic form*

$$\Phi(x) = (1 + \dot{T}_2(\varphi))\|\Omega_0^{T_2(\varphi)}(\varphi)x\|^2 - (1 + \dot{T}_1(\varphi))\|\Omega_0^{T_1(\varphi)}(\varphi)x\|^2$$

is negative definite: $\Phi(x) \leqslant -\gamma\|x\|^2$, then the system (12) is weakly regular. In addition, if the inequality $T_1(\varphi) < T_2(\varphi)$ is satisfied, the systems (2) and (12) are regular.

Let us recall $\|A\|_0 = \max\limits_{\varphi \in T_m} \|A(\varphi)\|$ for matrix $A = A(\varphi)$. Then the generalization of the theorem 2.5 is the following theorem.

Theorem 2.8. *Suppose that there exist $\Delta_i \in \mathbb{R}$, $i = 1, \ldots, k$, satisfying the inequality*

$$k \cdot \sum_{i=1}^{k} \|\Omega_0^{\Delta_i}(\varphi) \cdot P_i(\varphi)\|_0^2 < 1 \qquad \forall \varphi \in T_m, \tag{20}$$

where matrices $P_i(\varphi) \in C(T_m)$, $i = 1, \ldots, k$, fulfil the condition

$$\sum_{i=1}^{k} P_i(\varphi) \equiv I_n, \tag{21}$$

then the system (2) is regular.

Proof. Let us consider a symmetric matrix in the form

$$S(\varphi) = \sum_{i=1}^{k} S_i(\varphi) = \sum_{i=1}^{k} \int\limits_{-\Delta_i}^{0} \Omega_\tau^0(\varphi) P_i(\varphi_\tau(\varphi)) \cdot \left[\Omega_\tau^0(\varphi) P_i(\varphi_\tau(\varphi))\right]^T d\tau. \tag{22}$$

We show that if the inequality (20) is satisfied, the derivative of a quadratic form $\langle S(\varphi)x, x\rangle$ with respect to the system (12) is positive definite. Let us examine one of the element of the series (22):

$$S_i(\varphi) = \int\limits_{-\Delta_i}^{0} \Omega_\tau^0(\varphi) P_i(\varphi_\tau(\varphi)) \cdot \left[\Omega_\tau^0(\varphi) P_i(\varphi_\tau(\varphi))\right]^T d\tau,$$

and write down a composition

$$S_i(\varphi_t(\varphi)) = \int\limits_{-\Delta_i + t}^{t} \Omega_\tau^t(\varphi) P_i(\varphi_\tau(\varphi)) \cdot \left[\Omega_\tau^t(\varphi) P_i(\varphi_\tau(\varphi))\right]^T d\tau.$$

Differentiating with respect to variable t, we obtain

$$\begin{aligned}&\frac{d}{dt}S_i(\varphi_t(\varphi)) =\\&= P_i(\varphi_t(\varphi)) \cdot [P_i(\varphi_t(\varphi))]^T - \Omega^t_{t-\Delta_i}(\varphi)P_i(\varphi_{t-\Delta_i}(\varphi)) \cdot \left[\Omega^t_{t-\Delta_i}(\varphi)P_i(\varphi_{t-\Delta_i}(\varphi))\right]^T +\\&\qquad + A(\varphi_t(\varphi))S_i(\varphi_t(\varphi)) + S_i(\varphi_t(\varphi))A^T(\varphi_t(\varphi)),\end{aligned}$$

and hence for $t = 0$ we get

$$\begin{aligned}&\dot{S}_i(\varphi) - S_i(\varphi)A^T(\varphi) - A(\varphi)S_i(\varphi) =\\&\qquad = P_i(\varphi) \cdot [P_i(\varphi)]^T - \Omega^0_{-\Delta_i}(\varphi)P_i(\varphi_{-\Delta_i}(\varphi)) \cdot \left[\Omega^0_{-\Delta_i}(\varphi)P_i(\varphi_{-\Delta_i}(\varphi))\right]^T.\end{aligned}$$

In this way we obtain equality for the matrix (22):

$$\begin{aligned}&\dot{S}(\varphi) - S(\varphi)A^T(\varphi) - A(\varphi)S(\varphi) =\\&\quad = \sum_{i=1}^{k} P_i(\varphi) \cdot [P_i(\varphi)]^T - \sum_{i=1}^{k} \Omega^0_{-\Delta_i}(\varphi)P_i(\varphi_{-\Delta_i}(\varphi)) \cdot [\Omega^0_{-\Delta_i}(\varphi)P_i(\varphi_{-\Delta_i}(\varphi))]^T,\end{aligned}$$

hence the corresponding quadratic form is as follows

$$\begin{aligned}&\left\langle \left[\dot{S}(\varphi) - S(\varphi)A^T(\varphi) - A(\varphi)S(\varphi)\right]x, x\right\rangle =\\&\qquad = \sum_{i=1}^{k} \left\|[P_i(\varphi)]^T x\right\|^2 - \sum_{i=1}^{k} \left\|\left[\Omega^0_{-\Delta_i}(\varphi)P_i(\varphi_{-\Delta_i}(\varphi))\right]^T x\right\|^2. \qquad (23)\end{aligned}$$

Taking into account the identity (21), the first component of the right-hand side of the equation (23) is estimated from below in the following manner

$$\sum_{i=1}^{k} \|[P_i(\varphi)]^T x\|^2 \geqslant \frac{1}{k}\|x\|^2. \qquad (24)$$

Indeed, since $\sum\limits_{i=1}^{k} P_i(\varphi) \equiv I_n$, then

$$\|x\| = \left\|\sum_{i=1}^{k} [P_i(\varphi)]^T x\right\| \leqslant \sum_{i=1}^{k} \|[P_i(\varphi)]^T x\|.$$

Hence, based on the Cauchy–Schwarz inequality

$$\left(\sum_{i=1}^{k} a_i b_i\right)^2 \leqslant \left(\sum_{i=1}^{k} a_i^2\right) \cdot \left(\sum_{i=1}^{k} b_i^2\right),$$

by substituting $a_i = 1$, $b_i = \|[P_i(\varphi)]^T x\|$, we obtain

$$\|x\|^2 \leqslant \left(\sum_{i=1}^{k} \|[P_i(\varphi)]^T x\|\right)^2 \leqslant k \cdot \sum_{i=1}^{k} \|[P_i(\varphi)]^T x\|^2,$$

which implies the fulfilment of the inequality (24).

Now let us estimate the second component on the right-hand side of (23). To do this, let us consider the following inequality

$$\left\|\left[\Omega^0_{-\Delta_i}(\varphi)\,P_i\left(\varphi_{-\Delta_i}(\varphi)\right)\right]^T x\right\| \leqslant \left\|\left[\Omega^0_{-\Delta_i}(\varphi)\,P_i\left(\varphi_{-\Delta_i}(\varphi)\right)\right]^T\right\| \cdot \|x\| =$$
$$= \|\Omega^0_{-\Delta_i}(\varphi)\,P_i\left(\varphi_{-\Delta_i}(\varphi)\right)\|\cdot\|x\| \leqslant \max_{\varphi\in T_m} \|\Omega^0_{-\Delta_i}(\varphi)\,P_i\left(\varphi_{-\Delta_i}(\varphi)\right)\| \cdot \|x\| =$$
$$= \max_{\varphi\in T_m} \|\Omega^{\Delta_i}_0(\varphi)\,P_i(\varphi)\| \cdot \|x\| = \|\Omega^{\Delta_i}_0(\varphi)\,P_i(\varphi)\|_0 \cdot \|x\|.$$

Thus, the second term on the right-hand side of (23) is estimated as follows

$$\sum_{i=1}^{k} \left\|\left[\Omega^0_{-\Delta_i}(\varphi)\,P_i\left(\varphi_{-\Delta_i}(\varphi)\right)\right]^T x\right\|^2 \leqslant \sum_{i=1}^{k} \|\Omega^{\Delta_i}_0(\varphi)\,P_i(\varphi)\|_0^2\|x\|^2. \tag{25}$$

Taking into account inequalities (24), (25), from equation (23) follows

$$\left\langle\left[\dot S(\varphi) - S(\varphi)\,A^T(\varphi) - A(\varphi)\,S(\varphi)\right]x, x\right\rangle \geqslant \left(\frac{1}{k} - \sum_{i=1}^{k}\|\Omega^{\Delta_i}_0(\varphi)\,P_i(\varphi)\,\|_0^2\right)\cdot\|x\|^2.$$

This shows that if the inequality (20) is satisfied, then the derivative of the quadratic form $\langle S(\varphi)\,x, x\rangle$ with respect to the system (12) is positive definite, which means that the system (2) is weakly regular. □

Theorem 2.9 ([6]). *Let the system (2) be weakly regular, then the expanded system*

$$\frac{d\varphi}{dt} = a(\varphi), \qquad \frac{dx}{dt} = A(\varphi)x, \qquad \frac{dy}{dt} = x - A^T(\varphi)y, \tag{26}$$

is regular. Whereas the derivative of the following non-degenerated quadratic form

$$V_p = p\langle x, y\rangle + \langle S(\varphi)y, y\rangle, \tag{27}$$

with respect to the system (26) is positive definite for sufficiently large values of the parameter p (here the matrix $S(\varphi) \in C'(T_m; a)$ satisfies the inequality (13)).

Example 2.10. The system of equations $\frac{d\varphi}{dt} = \sin\varphi$, $\frac{dx}{dt} = (\cos\varphi)x$ is strictly weakly regular, because the derivative of the function $V = -(\cos\varphi)y^2$ with respect to the adjoint system: $\frac{d\varphi}{dt} = \sin\varphi$, $\frac{dy}{dt} = -(\cos\varphi)y$ is positive definite

$$\dot V = (\sin\varphi)\dot\varphi y^2 - (\cos\varphi)2y\dot y = (\sin^2\varphi + 2\cos^2\varphi)y^2 \geqslant y^2$$

whereas $\cos\varphi = 0$, $\varphi = \frac{\pi}{2} + \pi n$. The following expanded system

$$\frac{d\varphi}{dt} = \sin\varphi, \quad \frac{dx}{dt} = (\cos\varphi)x, \quad \frac{dy}{dt} = x - (\cos\varphi)y \tag{28}$$

is regular, because the derivative of the non-degenerated quadratic form

$$V = 2xy - (\cos\varphi)y^2 \tag{29}$$

with respect to the system (28) is positive definite

$$\dot{V} = 2x^2 - 2xy\cos\varphi + (\sin^2\varphi + 2\cos^2\varphi)y^2 \geqslant x^2 + y^2.$$

Remark 2.11. If the system (2) is weakly regular, then the expanded system

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x, \quad \frac{dy}{dt} = B(\varphi)x - A^T(\varphi)y, \tag{30}$$

is regular for each $n \times n$-dimensional matrix $B(\varphi) \in C(T_m)$ that satisfies one of the inequalities

$$\langle B(\varphi)x, x\rangle \geqslant \beta\|x\|^2, \quad \langle B(\varphi)x, x\rangle \leqslant -\beta\|x\|^2, \quad \beta = \text{const} > 0. \tag{31}$$

Whereas the Green-Samoilenko function of the system (30) will be $2n$-dimensional

$$\bar{G}_0(\tau,\varphi) = \begin{cases} \begin{pmatrix} \Omega_\tau^0(\varphi) & 0 \\ \omega(0,\tau,\varphi) & (\Omega_0^\tau(\varphi))^T \end{pmatrix} \begin{pmatrix} C_{11}(\varphi_\tau(\varphi)) & C_{12}(\varphi_\tau(\varphi)) \\ C_{21}(\varphi_\tau(\varphi)) & C_{22}(\varphi_\tau(\varphi)) \end{pmatrix}, & \tau \leqslant 0, \\ \begin{pmatrix} \Omega_\tau^0(\varphi) & 0 \\ \omega(0,\tau,\varphi) & (\Omega_0^\tau(\varphi))^T \end{pmatrix} \begin{pmatrix} C_{11}(\varphi_\tau(\varphi)) - I_n & C_{12}(\varphi_\tau(\varphi)) \\ C_{21}(\varphi_\tau(\varphi)) & C_{22}(\varphi_\tau(\varphi)) - I_n \end{pmatrix}, & \tau > 0 \end{cases}$$

and will be changing with the change of the matrix $B(\varphi) \in C(T_m)$. The n-dimensional block $G_0^{11}(\tau,\varphi)$ of the matrix $\bar{G}_0(\tau,\varphi)$ is the Green-Samoilenko function of the system (2). Still, there is no answer to the question whether all the Green-Samoilenko functions $G_0^{11}(\tau,\varphi)$ of the system (2) can be obtained from the system (30) by changing the matrix $B(\varphi) \in C(T_m)$.

Remark 2.12. If we resign from the conditions (31), the system (30) may not be regular. Whereas the question what necessary and sufficient conditions need to be imposed on the matrix $B(\varphi) \in C(T_m)$ so that the system (30) is regular under the condition of strictly weakly regularity of the system (2) remains open.

Continuing the study of the example (28), we will consider the following system

$$\frac{d\varphi}{dt} = \sin\varphi, \quad \frac{dx}{dt} = (\cos\varphi)x, \quad \frac{dy}{dt} = (\sin\varphi)x - (\cos\varphi)y. \tag{32}$$

Although the derivative of the quadratic form $V = pxy - (\cos\varphi)y^2$ with respect to the system (32) is not positive definite independently of the parameter $p \in \mathbb{R}$, a quadratic form can be written in another form

$$\Phi = x^2\cos\varphi + 2xy\sin\varphi - y^2\cos\varphi,$$

whose derivative with respect to the system (32) will be positive definite. It can be concluded that under the condition of strictly weakly regularity of the system (2), the expanded system (30) may be regular with some matrices $B(\varphi) \in C(T_m)$ that do not fulfil any of the inequalities (31).

One of the generalizations of the regular system of equations (32) is known. Consider the case $n = 1$, $A(\varphi) = \lambda(\varphi)$ is a continuous, 2π-periodic scalar function with respect to each variable φ_j, $j = 1, \dots, m$. Let us denote

$$\psi = k_1\varphi_1 + \ldots + k_m\varphi_m + \theta = \langle k, \varphi\rangle + \theta,$$

where k_j are certain integers, $k = (k_1, \ldots, k_m)$ is a vector with integer coordinates, $|k| = |k_1| + \ldots + |k_m|$, θ is a constant.

The following theorem is true.

Theorem 2.13 ([6]). *Let the inequality $\sigma = \langle k, a(\varphi)\rangle \sin\psi + 2\lambda(\varphi)\cos\psi > 0$ for some integer vector k, $|k| > 0$, for all φ_j, $j = 1, \ldots, m$, and for some constant θ be satisfied. Then the system of equations*

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = \lambda(\varphi)x, \quad \frac{dy}{dt} = \langle k, a(\varphi)\rangle - \lambda(\varphi)y, \tag{33}$$

is regular whereas the derivative of the quadratic form $x^2\cos\psi + 2xy\sin\psi - y^2\cos\psi$ with respect to the system (33) is positive definite.

Let us consider the following example (see [6]).

Example 2.14.

$$\begin{aligned}
&\frac{d\varphi_1}{dt} = 3\sin\varphi_1\cos\varphi_2, \qquad \frac{d\varphi_2}{dt} = 2\cos\varphi_1\sin\varphi_2,\\
&\frac{dx}{dt} = x\left[n\cos(\varphi_1 - \varphi_2) + \varepsilon\sin(\varphi_1 + \varphi_2)\right],
\end{aligned}$$

where $n = 1, 2, \ldots$, $|\varepsilon| < 0,5$. Denoting $\psi = \varphi_1 - \varphi_2$, $k = (1, -1)$, we have

$$\begin{aligned}
\langle k, a\rangle \sin\psi &= 2\sin^2\psi + \sin\varphi_1\cos\varphi_2\sin\psi,\\
2\lambda(\varphi)\cos\psi &= 2n\cos^2\psi + 2\varepsilon\cos\psi\sin(\varphi_1 + \varphi_2).
\end{aligned}$$

It is clear that the conditions of Theorem 2.13 given above are met. In this way, the expanded system of equations

$$\begin{aligned}
&\frac{d\varphi_1}{dt} = 3\sin\varphi_1\cos\varphi_2, \quad \frac{d\varphi_2}{dt} = 2\cos\varphi_1\sin\varphi_2,\\
&\frac{dx}{dt} = \left[n\cos(\varphi_1 - \varphi_2) + \varepsilon\sin(\varphi_1 + \varphi_2)\right]x,\\
&\frac{dy}{dt} = \left[3\sin\varphi_1\cos\varphi_2 - 2\cos\varphi_1\sin\varphi_2\right]x - \left[n\cos(\varphi_1 - \varphi_2) + \varepsilon\sin(\varphi_1 + \varphi_2)\right]y,
\end{aligned}$$

is regular for any natural value n and the fulfilment of the inequality $|\varepsilon| < 0,5$.

Let us consider the system of differential equations

$$\begin{aligned}
\frac{d\varphi}{dt} &= a\cos\varphi + b\sin\varphi,\\
\frac{dx}{dt} &= \Big(a_0 + \sum_{j=1}^{n}(a_j\cos j\varphi + b_j\sin j\varphi)\Big)x,
\end{aligned} \tag{34}$$

with some real coefficients a, b, a_j, b_i, $j = 0, \ldots, n$, $i = 1, \ldots, n$. The problem is to find such conditions for these coefficients that the system (34) has the Green-Samoilenko function.

It should be noted that if $a = b = 0$ in the system (34), it may have only one Green-Samoilenko function under the condition $a_0 + \sum_{j=1}^{n} (a_j \cos j\varphi + b_j \sin j\varphi) \neq 0$ for every $\varphi \in \mathbb{R}$.

Suppose that $a^2 + b^2 \neq 0$ and denote

$$\begin{aligned}
M_1 &= a_1 \cos\theta - b_1 \sin\theta + a_3 \cos 3\theta - b_3 \sin 3\theta + \dots \\
&\qquad + a_{2l-1} \cos(2l-1)\theta - b_{2l-1} \sin(2l-1)\theta, \\
M_2 &= a_0 + a_2 \cos 2\theta - b_2 \sin 2\theta + a_4 \cos 4\theta - b_4 \sin 4\theta + \dots \\
&\qquad + a_{2m} \cos 2m\theta - b_{2m} \sin 2m\theta,
\end{aligned}$$

where

$$\max\{2l-1, 2m\} = n, \quad \sin\theta = \frac{a}{\sqrt{a^2+b^2}}, \quad \cos\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

The following theorem is true.

Theorem 2.15 ([3]). *The system (34) has a unique Green-Samoilenko function whenever the inequality $|M_1| < |M_2|$ is satisfied. If the inequality $M_1 > |M_2|$ holds, then the system (34) has infinitely many different Green-Samoilenko functions. In the case of $|M_1| = |M_2|$, $M_1 < -|M_2|$, the system (34) has no Green-Samoilenko function.*

Theorem 2.16 ([8]). *Let the two systems*

$$\begin{cases} \frac{d\varphi}{dt} = \omega_1(\varphi), & \varphi \in T_m, \\ \frac{dx}{dt} = A_1(\varphi)x, & x \in \mathbb{R}^n, \end{cases} \qquad \begin{cases} \frac{d\psi}{dt} = \omega_2(\psi), & \psi \in T_k, \\ \frac{dx}{dt} = A_2(\psi)x, & x \in \mathbb{R}^n, \end{cases} \tag{35}$$

be weakly regular, then the following system

$$\begin{cases}
\frac{d\varphi}{dt} = \omega_1(\varphi), \\
\frac{d\psi}{dt} = \omega_2(\psi), \\
\frac{dx_1}{dt} = \left[A_2(\psi) + \frac{1}{2}(A_1(\varphi) + A_1^T(\varphi)) - I_n\right] x_1 + \\
\qquad + \left[A_1(\varphi) + A_2^T(\psi)\right] x_2, \qquad\qquad x_j \in \mathbb{R}^n, \\
\frac{dx_2}{dt} = \left[-A_2(\psi) + \frac{1}{2}(A_1(\varphi) - A_1^T(\varphi)) + I_n\right] x_1 - A_2^T(\psi) x_2 \\
\frac{dx_3}{dt} = \left[A_2(\psi) + \frac{1}{2}(A_1^T(\varphi) - A_1(\varphi)) + I_n\right] x_1 - \\
\qquad - \left[A_1(\varphi) + A_2^T(\psi)\right] x_2 - A_1^T(\varphi) x_3,
\end{cases} \tag{36}$$

is regular. In addition, the derivative of the quadratic form

$$V_p = p^2(\langle x_1, x_2\rangle + \langle x_1, x_3\rangle + \langle x_2, x_3\rangle) + p\langle S_2(\psi)x_2, x_2\rangle + \langle S_1(\varphi)x_3, x_3\rangle,$$

with respect to the system (28) for sufficiently large values of the parameter $p > 1$ is positive definite.

Corollary 2.17. *Let the system (2) be weakly regular, then the following system*

$$\begin{cases} \frac{d\varphi}{dt} = a(\varphi), \\ \frac{dx_1}{dt} = (\frac{3}{2}A(\varphi) + \frac{1}{2}A^T(\varphi) - I_n)x_1 + (A(\varphi) + A^T(\varphi))x_2, \quad x_j \in \mathbb{R}^n, \\ \frac{dx_2}{dt} = (-\frac{1}{2}A(\varphi) + \frac{1}{2}A^T(\varphi) + I_n)x_1 - A^T(\varphi)x_2, \\ \frac{dx_3}{dt} = (\frac{3}{2}A(\varphi) - \frac{1}{2}A^T(\varphi) + I_n)x_1 - (A(\varphi) + A^T(\varphi))x_2 - A^T(\varphi)x_3, \end{cases} \tag{37}$$

is regular whereas the derivative of the quadratic form

$$V_p = p^2(\langle x_1, x_2\rangle + \langle x_1, x_3\rangle + \langle x_2, x_3\rangle) + p\langle S(\varphi)x_2, x_2\rangle + \langle S(\varphi)x_3, x_3\rangle,$$

with respect to the system (37) for sufficiently large values of the parameter $p > 1$ *is positive definite.*

Remark 2.18. In the systems (36) and (37) the identity matrix can be replaced with the matrix $B(\varphi)$, which is positive definite and in this case the systems (36) and (37) remain regular.

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